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The Busemann–Petty problem via spherical harmonics

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Abstract

The Busemann–Petty problem asks whether symmetric convex bodies in $\mathbb{R}^n$ with smaller central hyperplane sections necessarily have smaller n-dimensional volume. The solution has recently been completed, and the answer is affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. In this article we present a short proof of the affirmative result and its generalization using the Funk–Hecke formula for spherical harmonics.

MSC: 52A20

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1. Introduction

The Busemann–Petty problem (BP-problem), posed in 1956 (see [BP]), asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$vol_{n-1}(K \cap H) \leqslant vol_{n-1}(L \cap H)$$

for every central hyperplane H in $\mathbb{R}^n$. Does it follow that

$$vol_n(K) \leqslant vol_n(L)?$$

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The answer is affirmative if $n \leqslant 4$ and negative if $n \geqslant 5$. The solution appeared as the result of a sequence of papers [Ba,Bo,G1,G2,Gi,GKS,K1,K2,Lu,LR,Pa,Z1,Z2] (see [Z2] or [GKS] for historical details).

The unified solution to the BP-problem for all dimensions from [GKS] is based on three main ingredients. The first is the concept of an intersection body introduced by Lutwak [Lu] in 1988. Let K and L be symmetric star bodies in $\mathbb{R}^n$. We say that K is the intersection body of L if the radius of K in every direction is equal to the $(n-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction, i.e. for every vector ξ from the unit sphere S^{n-1},

$$\rho_K(\xi) = vol_{n-1}(L \cap \xi^\perp), \tag{1}$$

where $\rho_K(x) = \max\{a \geqslant 0\colon ax \in K\}$ is the radial function of K and $\xi^\perp = \{x \in \mathbb{R}^n\colon (x,\xi) = 0\}$. A star body K in $\mathbb{R}^n$ is called an *intersection body of a star body* if there exists a star body L satisfying (1). A more general class of *intersection bodies* can be defined as the closure of intersection bodies of star bodies in the radial metric $d(K,L) = \sup_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|$. Lutwak [Lu] found the following connection between intersection bodies and the BP-problem (the original result of Lutwak was slightly improved in [G1] and [Z1]): if K is an intersection body then the answer to the BP-problem is affirmative for every L, and, on the other hand, if L is not an intersection body one can perturb it to construct a body K giving together with L a counterexample. Therefore, the answer to the BP-problem in $\mathbb{R}^n$ is affirmative if and only if every symmetric convex body in $\mathbb{R}^n$ is an intersection body.

The second component is the Fourier transform characterization of intersection bodies from [K1, Theorem 1]: a star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$. Here $\|x\|_K = \min\{a \geqslant 0\colon x \in aK\}$ is the *Minkowski functional* of K. Note that $\|x\|_K^{-1} = \rho_K(x)$.

The last step is a formula connecting the derivatives of parallel section functions with the Fourier transform of powers of the Minkowski functional. For a unit vector $\xi \in S^{n-1}$, the *parallel section function* of K in the direction of ξ is defined as a function on $\mathbb{R}$ given by

$$A_{K,\xi}(t) = vol_{n-1}(K \cap \{x \in \mathbb{R}^n\colon (x,\xi) = t\}), \ \ t \in \mathbb{R}.$$

It was shown in [GKS] that if K is origin-symmetric, infinitely smooth (i.e. $\|\cdot\|_K \in C^\infty(S^{n-1})$), and k is an even integer, then for every $\xi \in S^{n-1}$,

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{K,\xi}^{(k)}(0), \tag{2}$$

where $A_{K,\xi}^{(k)}$ stands for the derivative of the order k, and the Fourier transform is considered in the sense of distributions. If k is odd then

$$(\|x\|_K^{-n+k+1})^\wedge(\xi)$$

$$= (-1)^{(k+1)/2} 2(n-k-1)k! \int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - \cdots - A_{K,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\, dz. \tag{3}$$

A short proof of the Radon transform versions of these formulas was given in [BFM].

The unified solution to the BP-problem can now be explained as follows. First, a simple approximation argument reduces the problem to the case where K and L are infinitely smooth. By Brunn's theorem (see [S2]), if K is an origin-symmetric convex body then the central section is maximal in every direction, so the function $A_{K,\xi}$ has maximum at zero and $A''_{K,\xi}(0) \leqslant 0$ for every ξ. Putting $k = 2$ in (2) one concludes that the function $\|x\|_K^{-n+3}$ is a positive definite distribution for every origin-symmetric convex body K. Now if $n = 4$ we get that $\|x\|_K^{-1}$ is positive definite for every symmetric convex body K. This implies that every symmetric convex body in $\mathbb{R}^4$ is an intersection body and, by Lutwak's connection, the answer to the BP-problem in dimension 4 is affirmative. If the dimension $n = 5$ then to get the same conclusion one has to put $k = 3$ in (3). However, the third derivative of the parallel section function is not controlled by convexity, and one can easily construct a symmetric convex body K in $\mathbb{R}^5$ for which the right-hand side of (3) takes negative values for some ξ, so K is not an intersection body. This implies the negative answer to the BP-problem in $\mathbb{R}^5$.

The negative answer in dimensions 5 and higher leaves open a question of what must one know about the behavior of parallel section functions at zero to ensure the relation between the volumes. This question was answered in [K3]. Let K and L be $(k-1)$-smooth origin symmetric convex bodies in $\mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,

$$(-1)^{(k-1)/2} A_{K,\xi}^{(k-1)}(0) \leqslant (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0),$$

where k is an odd integer and $1 \leqslant k \leqslant n-1$. Then: (i) if $k \geqslant n-3$ we have $vol_n(K) \leqslant vol_n(L)$; (ii) if $k < n-3$ it is still possible that $vol_n(K) > vol_n(L)$.

In this article, we present a short proof of the affirmative part of the BP-problem and its generalization. This proof does not involve intersection bodies or the Fourier transform, we get the result as an immediate application of the Funk–Hecke formula for spherical harmonics.

2. Fractional derivatives and spherical harmonics

Let K be an infinitely smooth, origin symmetric convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$. The parallel section function of K in the direction of ξ can be written in the form $A_{K,\xi}(t) = \int_{(x,\xi)=t} \chi(\|x\|_K)\,dx$, where χ is the indicator function of the interval $[0,1]$.

Let $m \in \mathbb{N} \cup \{0\}$. For $q \in \mathbb{C}, -1 < Re(q) < m,\ q \neq 0, 1, \ldots, m-1$, the *fractional derivative of the order q* of the function $A_{K,\xi}$ at zero is defined by

$$A_{K,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(A_{K,\xi}(t) - A_{K,\xi}(0) - \cdots - A_{K,\xi}^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} \right) dt$$
$$+ \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} A_{K,\xi}(t)\,dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{A_{K,\xi}^{(k)}(0)}{k!(k-q)}. \tag{4}$$

It is easy to see that for a fixed q the definition does not depend on the choice of $m > Re(q)$, so the fractional derivatives $A_{K,\xi}^{(q)}(0)$ are correctly defined for all non-integer $q \in \mathbb{C}, Re(q) > -1$. Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C}\colon Re(q) > -1\}$ not including integers, and has simple poles at integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers, so after the division we get an analytic function in the whole domain $\{q \in \mathbb{C}\colon Re(q) > -1\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign:

$$A_{K,\xi}^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} A_{K,\xi}(t)|_{t=0}.$$

Denote by P_m the space of spherical harmonics of degree m on the unit sphere S^{n-1} in R^n. Recall that spherical harmonics of degree m are restrictions to the sphere of harmonic homogeneous polynomials of degree m. The dimension $N(n,m)$ of the space P_m can easily be computed. We consider P_m as a subspace of $L_2(S^{n-1})$. One can construct an orthonormal basis of the space $L_2(S^{n-1})$ consisting of spherical harmonics $Y_{m,j}$, where $m = 0, 1, 2, \ldots$, and for each fixed m the functions $Y_{m,j}$, $j = 1, 2, \ldots, N(n,m)$ form an orthonormal basis in P_m.

The Funk–Hecke formula [Mu, p. 20] is as follows: for every $Y_m \in P_m$, every continuous function f on $[-1, 1]$ and every $x \in S^{n-1}$,

$$\int_{S^{n-1}} f((x,\xi))\, Y_m(\xi)\, d\xi = \lambda(m)\, Y_m(x), \tag{5}$$

where $\lambda(m)$ is a constant given by

$$\lambda(m) = \frac{(-1)^m \pi^{(n-1)/2}}{2^{m-1}\Gamma(m + (n-1)/2)} \int_{-1}^{1} f(t) \frac{d^m}{dt^m} (1-t^2)^{m+(n-3)/2}\, dt. \tag{6}$$

In the case where $f(t) = |t|^q$, $q > -1$ we denote $\lambda(m) = \lambda_q(m)$. The numbers $\lambda_q(m)$ are easy to calculate. The next formula appeared in several places for different values of q (see [Gr, p. 103]; [K4,Ri,S1]).

Lemma 1. *If $q > -1$, $q \neq 2k$, $k \in N \cup \{0\}$, and $m \geq 0$ is an even integer then*

$$\lambda_q(m) = \frac{\pi^{n/2-1}\Gamma(q+1)\sin(\pi(m-q)/2)\Gamma((m-q)/2)}{2^{q-1}\Gamma((m+n+q)/2)}. \tag{7}$$

Proof. Assume first that $q > m$ and calculate the integral from (6) by parts m times. Then use the formula $\int_{-1}^{1} t^{2\alpha-1}(1-t^2)^{\beta-1}\, dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and two formulas for the Γ-function: $\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+1/2)/\pi^{1/2}$ and $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi x)$. We get (7) for $q>m$. Note that both sides of (7) are analytic functions of q in the domain $Re(q)>-1,\ q\neq 2k,\ k\in N\cup\{0\}$. By analytic extension, (7) holds for every q from this domain. $\square$

Lemma 2. *For every* $q\in\mathbb{C},\ -1<Re(q)<n-1$, $m=0,2,4,\ldots$ *and* $Y_m\in P_m$,

$$\int_{S^{n-1}} A_{K,\xi}^{(q)}(0)\, Y_m(\xi)\, d\xi$$
$$= \frac{2^{q+1}\pi^{n/2-1}\sin(\pi(m+q+1)/2)\Gamma((m+q+1)/2)}{(n-q-1)\Gamma((m+n-q-1)/2)}\int_{S^{n-1}} \|\theta\|_K^{-n+q+1}\, Y_m(\theta)\, d\theta. \quad (8)$$

Proof. Assume that $q\in(-1,0)$. Then, by the Fubini theorem,

$$\begin{aligned} A_{K,\xi}^{(q)}(0) &= \frac{1}{\Gamma(-q)}\int_0^\infty t^{-1-q}A_{K,\xi}(t)\, dt \\ &= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |t|^{-1-q}\left(\int_{(x,\xi)=t}\chi(\|x\|_K)\, dx\right) dt \\ &= \frac{1}{2\Gamma(-q)}\int_K |(x,\xi)|^{-1-q}\, dx. \end{aligned}$$

By the Funk–Hecke formula,

$$\begin{aligned} &\int_{S^{n-1}} A_{K,\xi}^{(q)}(0)\, Y_m(\xi)\, d\xi \\ &= \frac{1}{2\Gamma(-q)}\int_K \|x\|_2^{-1-q}\left(\int_{S^{n-1}}\left|\left(\frac{x}{\|x\|_2},\xi\right)\right|^{-1-q} Y_m(\xi)\, d\xi\right) dx \\ &= \frac{1}{2\Gamma(-q)}\lambda_{-1-q}(m)\int_K \|x\|_2^{-1-q}\, Y_m\left(\frac{x}{\|x\|_2}\right) dx \\ &= \frac{1}{2\Gamma(-q)}\lambda_{-1-q}(m)\int_{\mathbb{R}^n} \|x\|_2^{-1-q}\chi(\|x\|_K)\, Y_m\left(\frac{x}{\|x\|_2}\right) dx \\ &= \frac{1}{2\Gamma(-q)}\lambda_{-1-q}(m)\int_{S^{n-1}} Y_m(\theta)\, d\theta\int_0^\infty r^{n-q-2}\chi(r\|\theta\|_K)\, dr \\ &= \frac{1}{2\Gamma(-q)(n-q-1)}\lambda_{-1-q}(m)\int_{S^{n-1}} \|\theta\|_K^{-n+q+1}\, Y_m(\theta)\, d\theta, \end{aligned}$$

where $\|\cdot\|_2$ stands for the Euclidean norm. Using the expression for $\lambda_{-1-q}(m)$ from Lemma 1, we get the result for $-1<q<0$. Since both sides of (8) are analytic functions of q in the domain $\{q\in\mathbb{C}:\ -1<Re(q)<n-1\}$, the general statement of Lemma 2 follows by analytic continuation. $\square$

Lemma 3. *Let K, L be infinitely smooth, origin symmetric convex bodies in $\mathbb{R}^n$. Then, for every $q \in \mathbb{C}$, $-1 < Re(q) < n-1$,*

$$\int_{S^{n-1}} A_{K,\xi}^{(n-q-2)}(0) A_{L,\xi}^{(q)}(0)\, d\xi = \frac{2^n \pi^{n-2} \cos(\pi q/2) \cos(\pi(n-q-2)/2)}{(q+1)(n-q-1)} \int_{S^{n-1}} \|\theta\|_K^{-1-q} \|\theta\|_L^{-n+q+1}\, d\theta. \tag{9}$$

Proof. Since K and L are symmetric and infinitely smooth star bodies, $A_{K,\xi}^{(n-q-2)}(0)$ and $A_{L,\xi}^{(q)}(0)$ are even continuous functions of the variable ξ, so there are only even numbered terms in their spherical harmonics expansions in $L_2(S^{n-1})$ with respect to the basis $Y_{m,j}$. For every even integer $m \geqslant 0$, we can apply the formula of Lemma 2, first to the body L and the number q, and then to the body K and the number $n-q-2$ in place of q: for each $j = 1, \dots, N(n,m)$,

$$\int_{S^{n-1}} A_{L,\theta}^{(q)}(0)\, Y_{m,j}(\theta)\, d\theta \int_{S^{n-1}} A_{K,\theta}^{(n-q-2)}(0)\, Y_{m,j}(\theta)\, d\theta = \frac{2^n \pi^{n-2} \cos(\pi q/2) \cos(\pi(n-q-2)/2)}{(q+1)(n-q-1)} \times \int_{S^{n-1}} \|\theta\|_L^{-n+q+1}\, Y_{m,j}(\theta)\, d\theta \int_{S^{n-1}} \|\theta\|_K^{-q-1}\, Y_{m,j}(\theta)\, d\theta. \tag{10}$$

Since the functions $Y_{m,j}$ form an orthonormal basis in $L_2(S^{n-1})$, the result follows. $\square$

3. Results of the Busemann–Petty type

First, let us show that Lemma 3 provides an immediate proof of the affirmative answer to the BP-problem in the dimension 4. This result was first established by Zhang [Z2].

Theorem 1. *The answer to the BP-problem in $\mathbb{R}^4$ is affirmative.*

Proof. We can assume without loss of generality that the origin symmetric convex bodies K and L in the formulation of the BP-problem are infinitely smooth. Put $n = 4$ and $q = 0$ in the formula of Lemma 3. We get

$$\int_{S^3} A''_{K,\xi}(0) A_{L,\xi}(0)\, d\xi = -\frac{16\pi^2}{3} \int_{S^3} \|\theta\|_K^{-1} \|\theta\|_L^{-3}\, d\theta. \tag{11}$$

Replacing L by K in the latter equality, we get

$$\int_{S^3} A''_{K,\xi}(0)A_{K,\xi}(0)\,d\xi = -\frac{16\pi^2}{3}\int_{S^3} ||\theta||_K^{-4}\,d\theta. \tag{12}$$

It is given that for every $\xi \in S^{n-1}$,

$$A_{K,\xi}(0) = vol_{n-1}(K\cap\xi^\perp) \leqslant vol_{n-1}(L\cap\xi^\perp) = A_{L,\xi}(0).$$

By Brunn's theorem, since K is an origin-symmetric convex body, the central section is maximal in every direction, so the function $A_{K,\xi}$ has maximum at zero and $A''_{K,\xi}(0)\leqslant 0$ for every ξ. Therefore, the integral in the left-hand side of (11) is smaller than the integral in the left-hand side of (12). Using the polar formula for the volume and Hölder's inequality, we get

$$\begin{aligned} n\,vol_n(K) &= \int_{S^3} ||\theta||_K^{-4}\,d\theta \leqslant \int_{S^3} ||\theta||_K^{-1}||\theta||_L^{-3}\,d\theta \\ &\leqslant \left(\int_{S^3} ||\theta||_K^{-4}\,d\theta\right)^{1/4}\left(\int_{S^3} ||\theta||_L^{-4}\,d\theta\right)^{3/4} \\ &= (n\,vol_n(K))^{1/4}(n\,vol_n(L))^{3/4}. \end{aligned}$$

It immediately follows that $vol_n(K)\leqslant vol_n(L)$. $\square$

In a similar way, one can prove a more general fact, containing the affirmative part of the generalization to the BP-problem mentioned in the introduction.

Theorem 2. *Let K and L be infinitely smooth, origin symmetric convex bodies in $\mathbb{R}^n$, $n\geqslant 2$, and $q\in[n-4,n-1)$, $q>-1$, q is not an odd integer. Suppose that for every $\xi\in S^{n-1}$,*

$$\frac{A^{(q)}_{K,\xi}(0)}{\cos(\pi q/2)} \leqslant \frac{A^{(q)}_{L,\xi}(0)}{\cos(\pi q/2)}. \tag{13}$$

Then $vol_n(K)\leqslant vol_n(L)$.

Proof. Assume first that $q\neq n-3$. We prove that for every $\xi\in S^{n-1}$ and $q\in[n-4,n-1)$, $q\neq n-3$,

$$\frac{A^{(n-q-2)}_{K,\xi}(0)}{\cos(\pi(n-q-2)/2)} \geqslant 0.$$

The rest of the proof is the same as in Theorem 1.

Let $\beta = n-q-2$. Then $\beta\in(-1,2]$, $\beta\neq 1$. The case $\beta=0$ is trivial, and the case $\beta=2$ follows from Brunn's theorem, as in the proof of Theorem 1. If

$\beta \in (-1, 0)$ then

$$A_{K,\xi}^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} A_{K,\xi}(t)\, dt > 0,$$

and also $\cos(\pi\beta/2) > 0$. If $\beta \in (0, 2)$, $\beta \neq 1$ then it follows from the definition of the fractional derivative (use the fact that $A_{K,\xi}$ is an even function) that

$$A_{K,\xi}^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} (A_{K,\xi}(t) - A_{K,\xi}(0))\, dt. \tag{14}$$

By Brunn's theorem, for every t, $A_{K,\xi}(t) \leqslant A_{K,\xi}(0)$. On the other hand, the numbers $\cos(\pi\beta/2)$ and $\Gamma(-\beta)$ have opposite signs, which proves our claim.

Now we consider the case $q = n - 3$. In this case both sides of the formula of Lemma 3 vanish, because K is origin symmetric and $A_{K,\xi}$ is an even function. Divide both sides of (9) by $\cos(\pi(n - q - 2)/2)$, and let $q \to n - 3$. By (14) and Brunn's theorem,

$$\lim_{q \to n-3} \frac{A_{K,\xi}^{(n-q-2)}(0)}{\cos(\pi(n - q - 2)/2)} = -\frac{\pi}{2} \int_0^\infty t^{-2} (A_{K,\xi}(t) - A_{K,\xi}(0))\, dt \geqslant 0.$$

The details of this argument can be found in [GKS, p. 702]. Now the result of Theorem 2 follows from this limit version of Lemma 3. □

Remark 1. If $q \in [n - 4, n - 1)$, $q > -1$ is an odd integer, the result of Theorem 2 holds if one replaces condition (13) by the condition that, for every $\xi \in S^{n-1}$, the quantity

$$(-1)^{(q+1)/2} \int_0^\infty t^{-q-1} \left(A_{K,\xi}(t) - \sum_{j=0}^{(q-1)/2} \frac{t^{2j}}{(2j)!} A_{K,\xi}^{(2j)}(0) \right) dt \tag{15}$$

is less or equal than the same expression for the body L. This follows from the same limit argument as in the case $q = n - 3$ of Theorem 2.

Remark 2. The formula of Lemma 3 can be written in a different form using Theorem 2 from [GKS]. This formula turns into an equality of the Parseval type on the sphere:

$$\int_{S^{n-1}} (\|x\|_K^{-1-q})^\wedge(\xi) (\|x\|_L^{-n+q+1})^\wedge(\xi)\, d\xi = (2\pi)^n \int_{S^{n-1}} \|\theta\|_K^{-1-q} \|\theta\|_L^{-n+q+1}\, d\theta.$$

A quite technical independent proof of the latter formula can be found in [K3].

Remark 3. It follows from Lemma 3, that if the body K satisfies the condition $(-1)^{(n-q-2)/2}A_{K,\xi}^{(n-q-2)} \geqslant 0$ for every $\xi \in S^{n-1}$ then the statement of Theorem 2 holds true for any body L and any $q \in (-1, n-1)$ such that both numbers q and $n-q-2$ are not odd integers. If $n-q-2$ is an odd integer this condition on K can be replaced by the condition of Remark 1 with $n-q-2$ in place of q. It was shown in [K5, Theorem 4] that if q is an integer these conditions on K are equivalent to K being a $(q+1)$-intersection body (see the definition in [K5, p. 1508]). Therefore, we get a generalization of the first Lutwak's connection between intersection bodies and the BP-problem.

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